

Dynamics of weakly inhomogeneous oscillator populations: Perturbation theory on top of Watanabe-Strogatz integrability

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As has been shown by Watanabe and Strogatz (WS) [Phys. Rev. Lett., 70, 2391 (1993)], a population of identical phase oscillators, sine-coupled to a common field, is a partially integrable system: for any ensemble size its dynamics reduces to equations for three collective variables. Here we develop a perturbation approach for weakly nonidentical ensembles. We calculate corrections to the WS dynamics for two types of perturbations: due to a distribution of natural frequencies and of forcing terms, and due to small white noise. We demonstrate, that in both cases the complex mean field for which the dynamical equations are written, is close to the Kuramoto order parameter, up to the leading order in the perturbation. This supports validity of the dynamical reduction suggested by Ott and Antonsen [Chaos, 18, 037113 (2008)] for weakly inhomogeneous populations.

Dynamics of oscillator populations arouses large interest across different fields of science and engineering [1–3]. Relevant physical examples are arrays of Josephson junctions or lasers, metronomes on a common support, ensembles of electronic circuits, spin-torque, optomechanical and electrochemical oscillators [4], etc. The concept of coupled oscillator populations finds also broad application in life sciences, in particular in neuroscience [5], and even in description of social phenomena [6, 7]. The paradigmatic model in this field is the Kuramoto model of globally coupled phase oscillators [8, 9]. Remarkably, this setup and its simple generalizations explain not only the emergence of collective mode, which can be viewed as a nonequilibrium disorder-to-order transition [10], but also many other interesting dynamical phenomena like partial synchrony [11] and chimera states [12, 13].

A striking property of a system of N identical oscillators, sine-coupled to a common field (with the famous Kuramoto and the Kuramoto-Sakaguchi models being representatives of this class), is the partial integrability of the system for $N > 3$, established in the seminal work by Watanabe and Strogatz (WS) [14]. The WS theory (below we explain it in sufficient details) allows one to reduce the dynamics of N oscillators to that of three collective variables and $N - 3$ constants [15–17], see also recent review [3], and is valid for arbitrary common force, which can be, e.g., stochastic [18]. The identity of oscillators is essential, as well as the restriction that all units are forced equally. However, natural systems always possess at least a small degree of inhomogeneity, and the goal of this Letter is to extend the WS approach to cover this case.

To achieve this goal we first re-write the original equations of the inhomogeneous oscillator population in a equivalent, but suitable for a perturbation analysis, form. (This can be considered as an analogy to writing Hamil-

tonian equations in action-angle variables and thus bringing them to a form, ready for an approximate analysis.) Next, in the limit of small inhomogeneity we approximately reduce the dynamics to weakly perturbed WS equations with a certain distribution of WS constants. For an illustration of our theory we analyze two particular problems: (i) an ensemble with distributions of the natural frequencies and of the forcing, i.e. the case of a quenched (time-independent) disorder, and (ii) stochastic perturbations due to uncorrelated white-noise terms acting on all units. For both examples we derive in the thermodynamic limit $N \rightarrow \infty$ corrections to the WS equations as well as the relation between the Kuramoto order parameter and the WS complex amplitude in the leading order in the perturbation amplitude.

We start with general equations for N sine-coupled phase oscillators

$$\dot{\varphi}_k = \omega(t) + \text{Im} [H(t)e^{-i\varphi_k}] + F_k, \quad k = 1, \dots, N. \quad (1)$$

Here ω and H describe general time-dependent common forcing. In particular, H can depend on the mean field, like in the Kuramoto setup (see [19] for an example where mean field enters ω). F_k are general inhomogeneous terms that also can be time- and φ_k -dependent. Notice that the case $F_k = 0$ is solved by the WS theory.

We re-write Eq. (1) as

$$\frac{d}{dt} (e^{i\varphi_k}) = i[\omega(t) + F_k]e^{i\varphi_k} + \frac{1}{2}H(t) - \frac{e^{i2\varphi_k}}{2}H^*(t) \quad (2)$$

and perform the Möbius transformation [16] from φ_k to the WS complex amplitude z and new WS phases ψ_k , according to

$$e^{i\varphi_k} = \frac{z + e^{i\psi_k}}{1 + z^*e^{i\psi_k}}. \quad (3)$$

Next, we search for the solution for z in form of the WS equation (cf. Eq. (10) in [15]) with an additional complex perturbation term P , to be determined later:

$$\dot{z} = i\omega z + \frac{H}{2} - \frac{H^*}{2} z^2 + P. \quad (4)$$

Substituting Eqs. (1,3,4) into Eq. (2) we obtain for the WS phases

$$\dot{\psi}_k = \omega + \text{Im}(z^* H) + F_k \left[\frac{2\text{Re}(ze^{-i\psi_k}) + 1 + |z|^2}{1 - |z|^2} \right] - \frac{2\text{Im}[P(z^* + e^{-i\psi_k})]}{1 - |z|^2}. \quad (5)$$

Since the Möbius transformation (3) from φ_k to the set (ψ_k, z) is under-determined, we impose one complex condition to ensure uniqueness of determination of the complex variable z . Following Refs. [14–17] we require:

$$\frac{1}{N} \sum_{k=1}^N e^{i\psi_k} = 0. \quad (6)$$

If this condition is valid at $t = 0$ for the initial Möbius transformation, then it will be valid at all times provided $\frac{d}{dt} \sum_k \exp[i\psi_k] = 0$. Substituting here Eq. (5), we obtain

$$P - P^* U = \frac{i}{N} \sum_{k=1}^N F_k [z + (1 + |z|^2)e^{i\psi_k} + z^* e^{2i\psi_k}], \quad (7)$$

where $U = N^{-1} \sum_k \exp[i2\psi_k]$. Together with the complex conjugate of Eq. (7), this allows us to express P :

$$P = \frac{i}{(1 - |U|^2)N} \sum_{k=1}^N F_k [z(1 - Ue^{-2i\psi_k}) + (1 + |z|^2)(e^{i\psi_k} - Ue^{-i\psi_k}) + z^*(e^{2i\psi_k} - U)] . \quad (8)$$

Equations (4,5,8) constitute a closed system for new variables z, ψ_k . We emphasize that our derivation of Eqs. (4,5,8) is exact; the only restriction is that $|z| \neq 1$ and $|U| \neq 1$, i.e. the cases of full synchrony and of two perfect clusters are excluded.

As one can see from Eq. (8), for the non-perturbed case $F_k = 0$ the perturbation term P vanishes and one obtains the WS equations [14, 15]. In the standard formulation, for $F_k = 0$ we have $\dot{\psi}_k = \omega + \text{Im}(z^* H)$ and therefore one can introduce variable α , satisfying $\dot{\alpha} = \omega + \text{Im}(z^* H)$, and constants $\bar{\psi}_k = \psi_k - \alpha$, what completes the WS equations, see [15].

It is instructive to discuss physical meaning of the WS variables. The WS complex amplitude $z = \rho e^{i\Phi}$, $0 \leq \rho \leq 1$, is close to the standard Kuramoto order parameter Z defined as

$$Z = N^{-1} \sum_{k=1}^N e^{i\varphi_k}.$$

Indeed, substituting here the Möbius transformation (3), we obtain [15]

$$Z = N^{-1} \sum_{k=1}^N \frac{z + e^{i\psi_k}}{1 + z^* e^{i\psi_k}} = z - \sum_{m=2}^{\infty} (1 - |z|^2)(-1)^m (z^*)^{m-1} C_m, \quad (9)$$

where $C_m = N^{-1} \sum_k \exp[im\psi_k]$ are amplitudes of Fourier modes of the distribution of the WS phases ψ_k . Expression (9) is valid for $|z| < 1$. One can see that generally z deviates from Z , although their values coincide in the asynchronous state $z = Z = 0$. In the fully synchronous state one should use directly Eq. (3) which yields that for $|z| = 1$ either all phases φ_k coincide, i.e. also $|Z| = 1$, or at most one oscillator deviates from the fully synchronous cluster. A special case corresponds to the uniform distribution of the WS phases ψ_k , i.e. to the situation where all C_m with $m \neq kN$ vanish ($k \neq 0$ is an arbitrary integer). Then $z = Z$ in the thermodynamic limit $N \rightarrow \infty$, while for a finite N there are corrections $\sim |z|^N$. This allows one to consider the WS equation (4) as the one for the Kuramoto order parameter Z . For the Kuramoto-Sakaguchi problem, where $H = e^{i\beta} Z$, this yields the closed equation for the order parameter, first derived in a different way by Ott and Antonsen [20]. An essential part of our perturbation approach below deals with the distribution of the WS phases, in fact we show that due to inhomogeneities and noise it is close to the uniform one. A final remark on the meaning of the new variables: the third WS variable, angle α , determines shift of individual oscillators with respect to $\arg(Z)$ and is not important for the collective dynamics.

Analysis of Eqs. (4,5,8) is not an easy task, even under assumption that the perturbation terms F_k are small. The main difficulty is that generally the WS complex amplitude $z = \rho \exp[i\Phi]$ is time-dependent. Therefore below we restrict ourselves to the case where in the absence of perturbations $\rho = \text{const}$ and $\dot{\Phi} = \text{const}$. Such a regime with $\rho \neq 1$ appears at least in two situations that attracted large interest recently. The first one is the chimera state (see Refs. [12, 13]), where a part of the population is fully synchronous, while the other part is not. The latter sub-population is quantified by a uniformly rotating WS complex amplitude z , what implies $\rho = \text{const}$, $0 < \rho < 1$ [21]. Another situation is partial synchrony due to nonlinear coupling, described in Refs. [11]. In both cases, there exists a rotating reference frame where the WS complex amplitude is constant, i.e. $\rho \exp[i\Phi] = \text{const}$. In this frame, for the unperturbed state the quantities ω, H are constants and satisfy $H = -i2\omega\rho(1 + \rho^2)^{-1}e^{i\Phi}$. Below we consider perturbations to this state.

It is convenient to introduce shifted WS phases according to $\psi_k = \theta_k + \Phi$ and to write $P = (Q + iS)e^{i\Phi}$, then

the system (4,5,8) can be re-written as a system of real equations

$$\begin{aligned}\dot{\theta}_k &= \frac{F_k}{1-\rho^2} [2\rho \cos \theta_k + 1 + \rho^2] - \\ &\quad - 2 \frac{S\rho + S \cos \theta_k - Q \sin \theta_k}{1-\rho^2} + \Omega, \\ S &= \frac{1}{1-X^2-Y^2} \frac{1}{N} \sum_{k=1}^N F_k (2\rho \cos \theta_k + 1 + \rho^2) \\ &\quad \cdot (\cos \theta_k - X \cos \theta_k - Y \sin \theta_k), \\ Q &= \frac{-1}{1-X^2-Y^2} \frac{1}{N} \sum_{k=1}^N F_k (2\rho \cos \theta_k + 1 + \rho^2) \\ &\quad \cdot (\sin \theta_k - Y \cos \theta_k + X \sin \theta_k),\end{aligned}\quad (10)$$

where $\Omega = \omega \frac{1-\rho^2}{1+\rho^2}$ and $X + iY = N^{-1} \sum_k \exp[i2\theta_k]$. Formally, Eqs. (10) is a system of phase oscillators θ_k driven by forces F_k and subject to mean fields X, Y, S, Q . In the unperturbed case $F_k = 0$ it reduces to a system of uncoupled uniformly rotating phase oscillators.

Below we analyze Eqs. (10) for two types of perturbations. In the first setup we consider purely deterministic perturbations of the driving terms ω, H , namely we take $F_k = \varepsilon (u_k + \text{Im}[(f_k + ih_k)e^{i(\Phi - \varphi_k)}])$. Here u_k determine spreading of natural frequencies $\omega + \varepsilon u_k$, while the terms f_k, h_k describe variation of the forcing H for individual units (cf. Eq. (1)). Parameter ε explicitly quantifies the level of inhomogeneity of the system; in the following treatment it is assumed to be small. Expressing $\exp[-i\varphi_k]$ via the WS complex amplitude z and phases θ_k according to (3), we obtain

$$\begin{aligned}F_k &= \varepsilon \left[u_k + f_k \frac{2\rho + (\rho^2 - 1) \sin \theta_k}{2\rho \cos \theta_k + 1 + \rho^2} + \right. \\ &\quad \left. + h_k \frac{2\rho + (\rho^2 + 1) \cos \theta_k}{2\rho \cos \theta_k + 1 + \rho^2} \right],\end{aligned}$$

which should be substituted in (10).

We analyze the resulting system in the thermodynamic limit $N \rightarrow \infty$. In this limit the perturbation terms u_k, f_k, h_k are described by their distribution density $W(u, f, h)$ (without any restriction we can assume $\langle u \rangle = \langle f \rangle = \langle h \rangle = 0$); furthermore we seek for a solution with constant mean fields X, Y, S, Q . Then the system (10) can be solved self-consistently: we find the stationary distribution of θ_k for the given values of the perturbations $w(\theta|u, f, h)$, and then calculate the mean fields X, Y, S, Q according to

$$X = \iiint du df dh W(u, f, h) \int_0^{2\pi} d\theta w(\theta|u, f, h) \cos 2\theta, \quad (11)$$

and similarly for other quantities. Since the expressions are lengthy, we present only the sketch of the derivation.

One can see that the r.h.s. of the equation for $\dot{\theta}$ contains, together with constant terms, only terms $\sim \cos \theta, \sin \theta$, i.e. $\dot{\theta} = A + B \cos \theta + C \sin \theta$. Thus, the stationary distribution of the WS phases has the form $w(\theta|u, f, h) \sim (A + B \cos \theta + C \sin \theta)^{-1}$. As a result, the integrals over θ in Eq. (11) (and in similar expressions for Y, S, Q) are reduced to solvable integrals of the type $\int_0^{2\pi} d\theta \cos(n\theta + \theta_0) (A + B \cos \theta + C \sin \theta)^{-1}$. This leads to rather lengthy but exact expressions, that however can be expanded and simplified using the small parameter ε . The resulting formulas contain first and second powers of u, f, h , but the first powers disappear due to averaging with respect to density $W(u, f, h)$. The final formulas of the perturbation analysis, in the order $\sim \varepsilon^2$, are:

$$\begin{aligned}S &= -\frac{\varepsilon^2}{2\Omega(1-\rho^2)} [2\rho(1+\rho^2)\langle u^2 \rangle + \\ &\quad + (4\rho^2 + 2\rho(\rho^2 + 1)\langle u(f+h) \rangle + (1+\rho^2)^2\langle uh \rangle)], \\ Q &= -\frac{\varepsilon^2}{2\Omega} [(1+\rho^2)\langle uf \rangle + 2\rho\langle f^2 \rangle + 2\rho\langle hf \rangle], \\ X &= \frac{\varepsilon^2}{4\Omega^2(1-\rho^2)^2} [4\rho^2\langle u^2 \rangle + (\rho^2 + 1)^2\langle h^2 \rangle + \\ &\quad + 4\rho(\rho^2 + 1)\langle hu \rangle - (\rho^2 - 1)^2\langle f^2 \rangle], \\ Y &= \varepsilon^2 \frac{4\rho(\rho^2 - 1)\langle uf \rangle + (\rho^2 + 1)(\rho^2 - 1)\langle hf \rangle}{4\Omega^2(1-\rho^2)^2}.\end{aligned}\quad (12)$$

Let us consider two cases where these expressions simplify. If only the natural frequency of oscillators ω is distributed, but the force H is the same for all oscillators, then $u \neq 0, f = h = 0$. In this case

$$\begin{aligned}S &= -\frac{\varepsilon^2 \rho(1+\rho^2)}{\Omega(1-\rho^2)} \langle u^2 \rangle, \quad X = \varepsilon^2 \frac{\rho^2 \langle u^2 \rangle}{\Omega^2(1-\rho^2)^2}, \\ Q &= Y = 0.\end{aligned}\quad (13)$$

If the oscillators have the same frequency, but the sine-part of the force varies, then $u = h = 0$ and $f \neq 0$, and we have

$$Q = -\frac{\varepsilon^2 \rho \langle f^2 \rangle}{\Omega}, \quad X = -\frac{\varepsilon^2 \langle f^2 \rangle}{4\Omega^2}, \quad Y = S = 0. \quad (14)$$

It is instructive to see how such perturbations look in terms of the original Eq. (4) for the WS complex amplitude z ; this is accomplished by recalling that $P = (Q + iS)e^{i\Phi}$ and $z = \rho \exp[i\Phi]$, what yields

$$P = -i\varepsilon^2 \frac{z(1+|z|^2)}{(1-|z|^2)\Omega} \langle u^2 \rangle, \quad P = -\varepsilon^2 \frac{z}{\Omega} \langle f^2 \rangle, \quad (15)$$

for the two considered cases.

Finally, we express the Kuramoto order parameter $Z = \langle \exp[i\varphi] \rangle$ via the WS complex amplitude z , using general relation (9). For the inhomogeneous population we see that generally the amplitudes of the second harmonics

$C_2 = X + iY$ of this distribution are non-zero, $X, Y \sim \varepsilon^2$ (higher harmonics have higher orders in ε). Hence, the distribution of θ is non-uniform, though the corrections are small, $\sim \varepsilon^2$:

$$Z = \rho e^{i\Phi} [1 - (1 - \rho^2)(X + iY)],$$

what yields for the two considered cases, in the leading order,

$$\begin{aligned} Z &= z \left[1 - \frac{\varepsilon^2 |z|^2}{(1 - |z|^2)\Omega^2} \langle u^2 \rangle \right], \\ Z &= z \left[1 + \varepsilon^2 \frac{1 - |z|^2}{4\Omega^2} \langle f^2 \rangle \right]. \end{aligned} \quad (16)$$

As a second application of our approach, we consider noisy perturbations to the oscillators dynamics, taking $F_k = \varepsilon \xi_k(t)$, where $\xi_k(t)$ is a Gaussian white noise, $\langle \xi(t)\xi(t') \rangle = 2\delta(t - t')$. As above, we consider perturbation to the state with constant complex amplitude z , thus our starting point are Eqs. (10). Furthermore, we take the thermodynamic limit $N \rightarrow \infty$, allowing us to express the mean fields X, Y, S, Q as averages over the distribution of the WS phases. Here it is convenient not to average the equations for S, Q directly, but to find these mean fields from the solution of the Fokker-Planck equation for the distribution of the phase θ , which follows straightforwardly from the Langevin equation

$$\begin{aligned} \dot{\theta} &= \Omega - 2 \frac{\rho S}{1 - \rho^2} - \cos \theta \frac{S}{1 - \rho^2} + \sin \theta \frac{Q}{1 - \rho^2} + \\ &+ \varepsilon \xi(t) \left(\frac{1 + \rho^2 + 2\rho \cos \theta}{1 - \rho^2} \right). \end{aligned} \quad (17)$$

Looking for a stationary solution of the Fokker-Planck equation, we use smallness of ε and represent the stationary distribution density as $w = w_0 + \varepsilon^2 w_{2r} \cos 2\theta + \varepsilon^2 w_{2i} \sin 2\theta$ (the expansion starts from the second harmonics terms because of the condition $\langle \exp[i\theta] \rangle = 0$). This leads to the following expressions for S, Q, w_{2r}, w_{2i} :

$$\begin{aligned} S &= w_{2r} = 0, \\ Q &= -\varepsilon^2 \frac{2\rho(1 + \rho^2)}{1 - \rho^2}, \quad w_{2i} = \varepsilon^2 \frac{2\rho^2}{(1 - \rho^2)^2 \Omega} w_0. \end{aligned} \quad (18)$$

Again, it is instructive to express the result of the perturbation analysis in terms of Eq. (4) and to write a relation between the Kuramoto order parameter and the WS complex amplitude:

$$P = -\varepsilon^2 \frac{2z(1 + |z|^2)}{1 - |z|^2}, \quad Z = z \left[1 + \varepsilon^2 \frac{2i|z|^2}{(1 - |z|^2)\Omega} \right]. \quad (19)$$

Equations (15,16,19) are the main result of the perturbation theory. They provide a closed description of the nonideal populations of oscillators, where typically the driving field H explicitly depends on the Kuramoto order parameter Z .

We now discuss the results of the perturbation analysis. We have considered in details two situations, that have been previously treated in the framework of the standard WS analysis, i.e. for identical units. In both analyzed cases, for a non-identity of parameters of the oscillators and for a noisy driving, we obtained that the WS phases, which have an arbitrarily distribution in the non-perturbed case, tend to a nearly uniform distribution with corrections $\sim \varepsilon^2$. This results in the approximate relation between the Kuramoto order parameter and the WS complex mean field, which differ by a small deviation $\sim \varepsilon^2$. This means that for weakly perturbed situations, the WS equation can be used for the evolution of the Kuramoto order parameter, with account of above computed corrections $\sim \varepsilon^2$. As discussed above (see also [15]), the uniform distribution of the WS phases is the case where the WS equations reduce to the Ott-Antonsen equations [20]; sometimes this set of WS phases is called Ott-Antonsen manifold. Our perturbation analysis shows that small inhomogeneities “drive” the ensemble of oscillators to an ε^2 -vicinity of the Ott-Antonsen manifold, but not exactly to it.

A possible direction of future research is consideration of more generic perturbations. Of particular interest is the case when the interaction between units is more complex than pure sine-coupling, e.g., when the second harmonic terms $\sim \exp[-i2\varphi]$ are present. Such setup can also be treated in the framework of the developed perturbation analysis; the results will be reported elsewhere.

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